Written Homework #3

(1) $\begin{bmatrix}1\\1\\-1\end{bmatrix}x_1+\begin{bmatrix}0\\2\\3\end{bmatrix}x_2+\begin{bmatrix}t\\-3\\-7\end{bmatrix}x_3=b$ $\Big\}$ $\begin{bmatrix}1&0&t\\1&2&-3\\-1&3&-7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=b$ $\underset{R_2-R_1\to R_2}{\sim}$ $\begin{bmatrix}1&0&t\\0&2&-t-3\\-1&3&-7\end{bmatrix}$

$\underset{R_3+R_1\to R_3}{\sim}$ $\begin{bmatrix}1&0&t\\0&2&-t-3\\0&3&t-7\end{bmatrix}$ $\underset{R_3-\frac{3R_2}{2}\to R_3}{\sim}$ $\begin{bmatrix}1&0&t\\0&2&-t-3\\0&0&\frac{5t}{2}-\frac{5}{2}\end{bmatrix}$ $\Big\}$ This system will only have a unique solution if $\frac{5t}{2}-\frac{5}{2}\neq 0$, so if $t\neq 1$ (a)

$\begin{bmatrix}1&0&0\\1&2&0\\-1&3&0\end{bmatrix}$ $\Big\}$ From the first 2 rows, c_1 & c_2 both $=0$, so yes the first two vectors are linearly independent (b)

$a_4=\begin{bmatrix}1\\4\\-5\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&t&1\\1&2&-3&4\\-1&3&-7&-5\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=c$ $\Big\}$ Since there are more columns than rows, there isn't a unique solution for any value of t. (c)

(2) (a) $\left(\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right)+\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$ (b) Not possible. Adding another vector to the set would not change the fact that the others are scalar multiples of each other.

(c) spans $\mathbb{R}^3$ and is linearly independent must have exactly 3 vectors. Spans $\mathbb{R}^3$ and is linearly dependent has at least 4 vectors. Doesn't span $\mathbb{R}^3$ but is linearly independent means it has at most 2 vectors. Doesn't span $\mathbb{R}^3$ and is linearly dependent has at most 3 vectors.

(e) It is not possible to have ∞ solutions by adding one more equation, since the other parts of the system already intersect at a unique solution.

(f) Deleting an inconsistent entry of a system that would otherwise have ∞ solutions would work. For example,

$\begin{cases}x+y=12\\2x+2y=24\\x+y=4\end{cases}$ $\leftarrow$ $\Big\{$ Deleting the last entry in this system of equations would give the system ∞ solutions.

(3) (a) The possible values of n for which the vectors u_1, u_2, u_3 can span $\mathbb{R}^3$ is 3.

(b) $\{u_1, u_1+u_2, u_1-u_3\}$ does span $\mathbb{R}^n$ because it is a linear combination of $\{u_1,u_2,u_3\}$ which already span $\mathbb{R}^n$

(c) False. e.g: $\left(\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$ is L.I., $\left(\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\2\end{bmatrix}\right)$ is not.

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(3) (d) $\{u_1, u_1+u_2, u_1-u_3\}$ is linearly independent because $a=0, b=0, c=0$ for the equation $au_1+bu_2+cu_3=0$, so for the equation $au_1+b(u_1+u_2)+c(u_1-u_3)=0$, $a, b, \& c$ must also $=0$.

(e) Since u_1, u_2, u_3 do not span $\mathbb{R}^3$, they must be linearly dependent, and therefore their plane equation can be found by finding that linear combination & plugging it into $ax+by+cz=0$. The plane would pass through the origin.

(4) $B=\begin{bmatrix} 3 & 0 & -1 & 5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (a) $n=3$ because it is the dimension of the matrix / the number of rows

(b) $M=4$ because it is the number of vectors / number of columns

(c) Since there is a row of all 0's at the bottom, the vectors are linearly dependent

(d) The set does not span $\mathbb{R}^n$ since there is a row of all 0's at the bottom.

(e) $\{u_1, u_3\}$ would be a linearly independent subset since those are pivot columns

(f) No, since all of B doesn't span $\mathbb{R}^n$ there isn't a subset that could span it either

(g) $B\vec{x}=\vec{b}$ does not have a solution, if $b=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, since that would imply $0=1$ in the last row which is not possible

(i) There is no unique solution for $B\vec{x}=\vec{b}$ since the last row of all 0's means there will either be ∞ solutions or no solutions (if you add a nonzero to the last row).

(h) The vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ can be added, to give the system ∞ solutions

(j) Yes, a linearly independent vector could be added to B to guarantee it spans all of $\mathbb{R}^n$, so $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ could be added.

(k) Yes, the zero vector in the second column is in the span of all the rest

(l) Other than the zero vector, there is not another column that is in the span of all the others.

(4) (m) $\begin{bmatrix} 3 & 0 & -1 & 5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \frac{1}{3}R_1 \\ \frac{1}{2}R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{1}{3}R_2 + R_1 \to R_1 \sim \begin{bmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(n) $\left. \begin{aligned} x_4 &= s_1 \\ x_3 &= \tfrac{1}{2} - s_1 \\ x_2 &= 0 \\ x_1 &= -\tfrac{3}{2} s_1 \end{aligned} \right\}$ general solution $\Rightarrow$ Nonzero solution $s_1 = 1 \Rightarrow \begin{bmatrix} x_1 = -\frac{3}{2} \\ x_2 = 0 \\ x_3 = -\frac{1}{2} \\ x_4 = 1 \end{bmatrix}$

(o) 2 free variables (since there are two columns without leading entries

(p) No, the matrix would not become linearly independent because the number of columns would still be greater than the number of rows.

(q) Yes, you could add rows that create a matrix in echelon form, for example adding $0x_1 + 2x_2 + 0x_3 + 0x_4$ would create a matrix of: $\begin{bmatrix} 3 & 0 & -1 & 5 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(5) (a) The system has 1 unique solution: $(0,0)$

(b) The smallest number of equations is 2: $y=0$ & $x=0$

Linear system 1: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Linear System 2 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) Nothing happens to the ~~∞~~ linear system, but the matrix would change as ~~another~~ row of zeroes is added.

(d) The system would become inconsistent because $ax+by=0$, so if it also $= 0.00001$, then $0 = 0.00001$ which is impossible

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(1) Let $\mathbf{a_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{a_2} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, and $\mathbf{a_3} = \begin{bmatrix} t \\ -3 \\ -7 \end{bmatrix}$.

(a) Find all values of t (if any) for which there will be a unique solution to $\mathbf{a_1}x_1 + \mathbf{a_2}x_2 + \mathbf{a_3}x_3 = \mathbf{b}$ for every vector $\mathbf{b}$ in $\mathbb{R}^3$. Explain your answer.

(b) Are the vectors $\mathbf{a_1}$ and $\mathbf{a_2}$ from part (a) linearly independent? Explain your answer.

(c) Let $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{a_3}$ be as in (a). Let $\mathbf{a_4} = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$. Without doing any further calculations, find all values of t (if any) for which there will be a unique solutions to $\mathbf{a_1}y_1 + \mathbf{a_2}y_2 + \mathbf{a_3}y_3 + \mathbf{a_4}y_4 = \mathbf{c}$ for every vector $\mathbf{c}$ in $\mathbb{R}^3$. Explain your answer.

(2) For each of the situations described below, **give an example** (if it's possible) or **explain why it's not possible**.

(a) A set of vectors that does not span $\mathbb{R}^3$. After adding one more vector, the set does span $\mathbb{R}^3$.

(b) A set of vectors that are linearly dependent. After adding one more vector, the set becomes linearly independent.

(c) A set of vectors in $\mathbb{R}^3$ with the following properties (four possibilities):

spans $\mathbb{R}^3$, linearly independent	spans $\mathbb{R}^3$, linearly dependent
doesn't span $\mathbb{R}^3$, linearly independent	doesn't span $\mathbb{R}^3$, linearly dependent

For each case that is *possible*, how many vectors could be in the set? (State any constraints, as in "there must be at least..." or "at most...")

(e) A system of equations with a unique solution. After adding another equation to the system, the new system has infinitely-many solutions.

(f) A system of equations without any solutions. After deleting an equation, the system has infinitely-many solutions.

(3) In each of the following cases, either find an example that contradicts the statement showing that it is false, or explain why the statement is always true.

(a) If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a spanning set for $\mathbb{R}^n$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is also a spanning set for $\mathbb{R}^n$. What are all possible values of n for which three vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ can span $\mathbb{R}^n$?

(b) If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a spanning set for $\mathbb{R}^n$, then $\{\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_3\}$ also spans $\mathbb{R}^n$.

(c) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent, then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are also linearly independent.

(d) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent, then $\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_3$ are also linearly independent.

(e) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ do not span $\mathbb{R}^3$, then there is a plane P in $\mathbb{R}^3$ that contain all of them. (Bonus: how can we find this plane? Does the plane go through the origin?)

(4) Recall that if we have m vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ in $\mathbb{R}^n$, then we can form the matrix A whose columns are $\mathbf{u}_1, \ldots, \mathbf{u}_m$. Let B be the echelon form of A. All the questions

below are based on such a matrix B. Most questions have a yes/no answer. Give full reasons for all answers.

Suppose we are given the following matrix B:

$$\begin{bmatrix} 3 & 0 & -1 & 5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the columns of B are not $\mathbf{u}_1, \ldots, \mathbf{u}_m$.

(a) What is n?
(b) What is m?
(c) Are $\mathbf{u}_1, \ldots, \mathbf{u}_m$ linearly independent?
(d) Does $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ span $\mathbb{R}^n$?
(e) Looking at B can you write down a subset of the original set $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ that would be guaranteed to be linearly independent?
(f) Is there a subset of the original set $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ that would be guaranteed to span $\mathbb{R}^n$?
(g) Write down a $\mathbf{b} \in \mathbb{R}^n$ for which $B\mathbf{x} = \mathbf{b}$ does not have a solution.
(h) Write down a $\mathbf{b} \in \mathbb{R}^n$ for which $B\mathbf{x} = \mathbf{b}$ has a solution.
(i) Write down a $\mathbf{b} \in \mathbb{R}^n$ for which $B\mathbf{x} = \mathbf{b}$ has a unique solution.
(j) Is there a new vector $\mathbf{w} \in \mathbb{R}^n$ that you could add to the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ to guarantee that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}\}$ will span $\mathbb{R}^n$?
(k) Is there a column of B that is in the span of the rest? If so, find it.
(l) Looking at B do you see a $\mathbf{u}_i$ that is in the span of the others? How can you identify it?
(m) Put B into reduced echelon form.
(n) Write down a non-zero solution of $A\mathbf{x} = \mathbf{0}$ if you can.
(o) How many free variables are there in the set of solutions to $A\mathbf{x} = \mathbf{b}$ when there is a solution?
(p) If you erased the last row of zeros in B then would the columns of the resulting matrix be linearly independent?
(q) Can you add rows to B to make the columns of the new matrix linearly independent? If yes, give an example of the new matrix you would construct.

(5) Consider the infinite system of linear equations in two variables given by $ax + by = 0$ where (a, b) moves along the unit circle in the plane.

(a) How many solutions does this system have?
(b) What is the smallest number of equations in the above system that have the same solution set? Write down two separate such linear systems, in vector form.
(c) What happens to the infinite linear system if you add the equation $0x + 0y = 0$ to it?
(d) What happens to the infinite linear system if by accident one of the equations was recorded as $ax + by = 0.00001$?

Explain all your answers in words.